Physics 195 / Applied Physics 195 — Assignment #10

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Due: 4:50 pm + 10 min grace period, Dec. 19, 2017 at the dropbox outside Maxwell-Dworkin Room 131.

Problem 1 (150 pt; no collaboration): Plasmonic excitation in 3D conductor

This problem seeks to guide you through the concept and technical details of Lecture #20. The plasmonic excitation in 3D conductor can be described by solving for the local electron density $n(\vec{r},t) = n_0 + \delta_n(\vec{r},t)$ and local electron velocity $\vec{v}(\vec{r},t)$, where n_0 is the equilibrium electron density (whose charge is balanced by the same density of background positive ions). Dynamical equations are:

$$m\frac{d\vec{v}}{dt} = -e\vec{E} - \frac{1}{n}\vec{\nabla}P; \quad (1)$$

$$\frac{\partial n}{\partial t} = -\vec{\nabla}\cdot(n\vec{v}), \quad (2)$$

where m is a short-hand notation for the collective mass per electron. Eq. (2) is the continuity equation stating the electron number conservation. Eq. (1) is the equation of motion, whose left hand side represents the inertial acceleration and right hand side consists of Coulomb and Pauli restoring forces. The Coulomb force $-e\vec{E}$ arises due to the electron density perturbation δ_n that results in charge imbalance (see Eq. (5)). The Pauli force arises due to the electron density perturbation δ_n that results in chemical potential imbalance, or the gradient in electron degeneracy pressure P. Using $\vec{\nabla}P = (\partial P/\partial n)_{n_0}\vec{\nabla}n$ and assuming that δ_n and $\vec{v}$ are small perturbations from equilibrium, Eqs. (1) and (2) may be linearized to

$$\frac{\partial\vec{v}}{\partial t} = -\frac{e}{m}\vec{E} - \frac{\alpha}{n_0}\vec{\nabla}\delta_n; \quad (3)$$

$$\frac{\partial\delta_n}{\partial t} = -n_0\vec{\nabla}\cdot\vec{v}, \quad (4)$$

where $\alpha \equiv (1/m)\,(\partial P/\partial n)_{n_0} = (3/5)v_F^2$ (v_F: Fermi velocity).[1] In addition to these two dynamical equations, we need Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = -\frac{e\delta_n}{\epsilon_0} \quad (5)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad (6)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad (7)$$

$$\vec{\nabla}\times\vec{B} = -\mu_0 n_0 e\vec{v} + \epsilon_0\mu_0\frac{\partial E}{\partial t} \quad (8)$$

Eq. (5) is a vital self-consistency statement that connects the local electron density perturbation δ_n to the local Coulomb restoring field $\vec{E}$, which is used in Eq. (3). Eq. (8) provides another connection—in addition to Eq. (3)—between $\vec{E}$ and $\vec{v}$.

(a) **Longitudinal plasmonic mode (Langmuir wave)**

We first solve for the longitudinal mode, where $\vec{E}$ is parallel to the wave vector $\vec{k}$, *i.e.*, $\vec{\nabla}\times\vec{E} = 0$.

- Under $\vec{\nabla}\times\vec{E} = 0$ ($\vec{k} \parallel \vec{E}$), show $\vec{v} \parallel \vec{E} \parallel \vec{\nabla}\delta_n$. Notice that $\vec{B} = 0$ under $\vec{\nabla}\times\vec{E} = 0$, which means that this longitudinal mode cannot give rise to an electromagnetic radiation.

[1] This calculation of α is not trivial: see Section F, Lecture #20.

- Under $\vec{\nabla} \times \vec{E} = 0$, show that δ_n and $\vec{v}$ are described by the following wave equations:

$$\frac{\partial^2 \delta_n}{\partial t^2} + \omega_p^2 \delta_n - \alpha \nabla^2 \delta_n = 0; \quad (9)$$

$$\frac{\partial^2 \vec{v}}{\partial t^2} + \omega_p^2 \vec{v} - \alpha \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = 0. \quad (10)$$

where $\omega_p = (n_0 e^2/m\epsilon_0)^{1/2}$ is the plasma frequency. Using harmonic analysis, show that the dispersion relation of this longitudinal mode is given by

$$\omega^2 = \omega_p^2 + \alpha k^2 \qquad \text{(Bohm-Gross dispersion)} \quad (11)$$

This longitudinal mode is the plasmonic wave where the electron density is spatiotemporally modulated according to Eq. (9).

(Note 1) The formalism above applies to not only conductors with dense population of electrons, but also classical plasma with rarefied electrons (*e.g.*, ionosphere, solar wind, intergalactic medium, and many more). In the classical plasma, m is the intrinsic electron mass, n_0 is typically small and yields a low plasma frequency ω_p, and α arises not from electron degeneracy pressure but from thermal pressure. In classical plasma, α due to thermal pressure is proportional to temperature. When α cannot be ignored at high enough temperature, the classical plasma is called 'warm.' If α is ignorable at low enough temperature, the classical plasma is 'cold.' In conductors, α due to electron degeneracy pressure is not temperature dependent to the first order.

(Note 2) If we ignore the Pauli restoring force (set α to 0), Eqs. (9) and (10) collapse to time differential equations with no spatial derivatives:

$$\frac{\partial^2 \delta_n}{\partial t^2} + \omega_p^2 \delta_n = 0; \quad (12)$$

$$\frac{\partial^2 \vec{v}}{\partial t^2} + \omega_p^2 \vec{v} = 0. \quad (13)$$

These correspond to the longitudinal bulk oscillation at a single frequency ω_p discussed in Section B of Lecture #20 and Problem of Homework #2. This bulk oscillation is not a wave (*i.e.*, there is no spatial modulation), or it is a wave with infinite wavelength. Thus you can appreciate that the Pauli restoring force is vital to create the longitudinal plasmonic wave in a 3D conductor while the Coulomb restoring force alone only creates the bulk oscillation (this is not the case with 2D conductors, where Coulomb restoring force alone without Pauli restoring force can create plasmonic waves; see Problem 2).

(b) **Transverse electromagnetic mode**

We now solve for a transverse mode, where $\vec{E} \perp \vec{k}$, that is, $\vec{\nabla} \cdot \vec{E} = 0$.

- Under $\vec{\nabla} \cdot \vec{E} = 0$ ($\vec{k} \perp \vec{E}$), show $\vec{k} \perp \vec{B}$ and $\vec{E} \perp \vec{B}$ while $\vec{E} \parallel \vec{v}$ and $\delta_n = 0$. This is a transverse electromagnetic wave, whose electric field grabs and moves electrons perpendicularly to the wave propagation direction. Spatiotemporal electron density modulation does not occur.
- Under $\vec{\nabla} \cdot \vec{E} = 0$, show that the local electron velocity $\vec{v}$ is described by the following wave equation:

$$\frac{\partial^2 \vec{v}}{\partial t^2} + \omega_p^2 \vec{v} - c^2 \nabla^2 \vec{v} = 0. \quad (14)$$

Using harmonic analysis, show that the dispersion relation for this transverse mode is given by

$$\omega^2 = \omega_p^2 + c^2 k^2. \quad (15)$$

This transverse mode is the electromagnetic mode that propagates through the conductor, if the incoming electromagnetic wave satisfies $\omega > \omega_p$ ('ultraviolet transparency' for metals). If $\omega < \omega_p$ for which k becomes imaginary, the electromagnetic wave radiated onto the conductor is reflected. This formalism applies again also to classical plasma; reflecting a radio wave off of the ionosphere with $\omega < \omega_p$ is a well-known example.

(c) **General approach that subsumes the results of Parts (a) and (b)**

We can study the collective electron excitation without *a priori* assuming any particular mode (*e.g.*, longitudinal or transverse) of excitation. By applying harmonic analysis to Eqs. (3-8) generally, show that

$$(\omega^2 - \omega_p^2 - \alpha k^2)\vec{E}_{\parallel} + (\omega^2 - \omega_p^2 - c^2 k^2)\vec{E}_{\perp} = 0 \tag{16}$$

where $\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp}$, $\vec{k} \parallel \vec{E}_{\parallel}$, and $\vec{k} \perp \vec{E}_{\perp}$. From this general secular equation, argue that Eq. (11) and (15) naturally emerge as dispersion relations for the longitudinal and transverse mode.

(d) **Wave propagation speed and dielectric constant**

Calculate the wave propagation velocity (phase velocity) v_L and v_T—not to be confused with the local electron velocity $\vec{v}$—for the longitudinal plasmonic wave and the transverse electromagnetic wave as functions of ω. From this, determine the frequency-dependent dielectric constants for each wave.

Problem 2 (150 pt): Plasmonic excitation in gated 2D conductor

Consider a 2D conductor that is placed in parallel with a metallic plate in close proximity at a small distance[2] d. The equilibrium conduction electron density per unit area in the 2D conductor is n_0 and the collective mass per electron in the 2D conductor is simply denoted as m. Ignoring the Pauli restoring force (that is, considering only the Coulomb restoring force), derive the wave equation for the per-unit-area electron density perturbation δ_n and from this show that the dispersion relation for the plasmonic wave that propagates on the 2D conductor is given by

$$\omega = \sqrt{\frac{n_0 e^2}{mC}}\,k, \tag{17}$$

where C is the electrostatic capacitance (per unit area) between the 2D conductor and the metallic plate. Note that the plasmonic velocity v_p (which corresponds to v_L of the 3D case of Problem 1) is then given by

$$v_p = \sqrt{\frac{n_0 e^2}{mC}}. \tag{18}$$

Problem 3 (100 pt): Charge screening in degenerate electron gas

An external point charge Q is inserted into a metal whose conduction electron density is n_0 and chemical potential is μ_0. The electrostatic dielectric constant due to bound electrons is κ. Calculate the overall steady state potential (screened potential) as a function of position $\vec{r}$, which is established after inserting this charge, by using the self-consistent field method as found in Lecture #21. Calculate the characteristic charge screening length (Thomas-Fermi screening length) for copper, gold, and silver.

[2] Rigorously, this close proximity is defined with $d \ll \lambda$ where λ is plasmonic wavelength of concern, but you don't have to worry too much about this here.